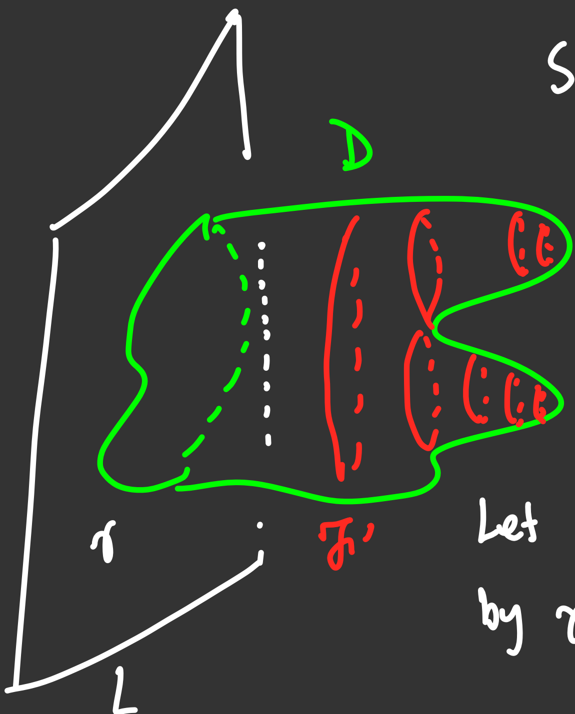
M closed oriented 3-manifold

$\mathcal{F}$ foliation by subsurfaces.

<u>Claim</u>: $\mathcal{F}$ taut $\Rightarrow L \subseteq M$ is incompressible $\forall L$ leaves of $\mathcal{F}$.



Suppose $\gamma \subseteq L$ not nullhomotopic.

But γ nullhomotopic inside M

Let $D \subseteq M$ disk bounded by γ in M.

Let's do Morse theory on D with the leaves of $\mathcal{F}$ instead of level sets of a function.

$\mathcal{F}' := \mathcal{F} \cap D$ "characteristic foliation" on D

We may perturb D to be generic wrt $\mathcal{F}$ s.t.

$\mathcal{F}'$ consists of some local max/min/saddle.

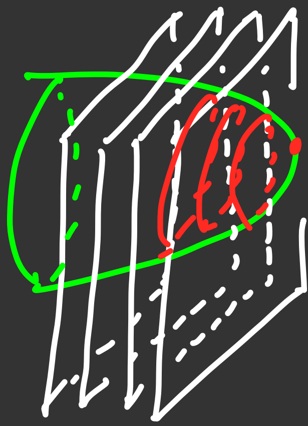
$\mathcal{F}'$ must contain a local max/min since,

$$1 = \chi(D^2) = \#\max - \#\text{saddle} + \#\min$$

by Poincaré-Hopf.

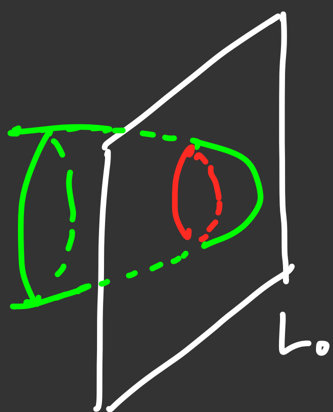
In a small nbhd of such a max/min,

the picture must look as follows -

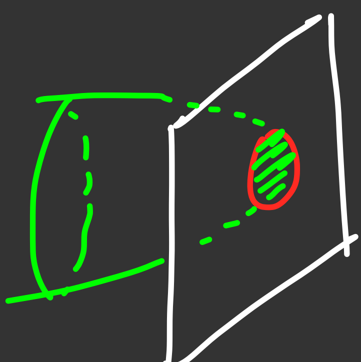


note:

red circular leaf $L_0 \cap D$ of $\mathcal{F}'$ bounds a disk in L_0 as well.



"push in" the green disk D



if we can keep pushing, we would be able to homotope D to a disk in $L \supset \partial D$, fixing $\partial D = \gamma$

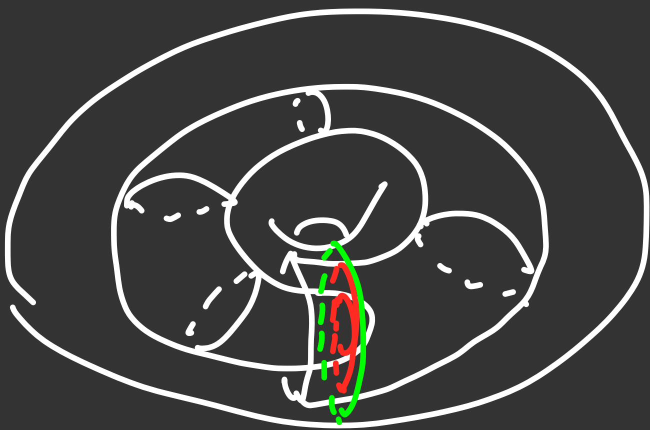
$\Rightarrow$ γ would bound a disk in L.

$\therefore$ $\ker(\pi_1 L \longrightarrow \pi_1 M)$ would be trivial, done.

- What obstructs us from pushing inward?

(a) Further critical points in D other than the max/min we started pushing from.

(b) Even if $\exists$ unique max/min, the following can occur:



$M = S^1 \times D^2$ (solid torus)

$\mathcal{F}$ = Reeb foliation

D = meridian disk (green)

Lemma: (b) cannot occur if $\mathcal{F}$ is taut.

We need the following theorem:

<u>Thm (Sullivan)</u>: $\mathcal{F}$ is a taut foliation on M

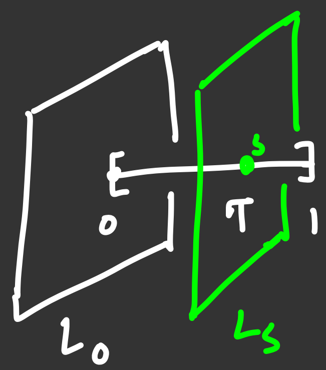
$\Leftrightarrow$ $\exists$ a metric g on M s.t. the leaves L of $\mathcal{F}$ are minimal surfaces in M.

Pf of Lemma:

Suppose $L_0 \subseteq M$ is a leaf of $\mathcal{F}$

ℓ_0 a component of $D \cap L_0$, a leaf of $\mathcal{F}'$

$T = [0, 1]$ be a transverse arc to the foliation with parameter $0 \le s \le 1$ s.t. $s = 0$ lies in L_0.



L_s be the leaf passing through the point $s = s$ on T.

Let ℓ_s be components of $D \cap L_s$ (hence leaves of $\mathcal{F}'$) s.t. $\{\ell_s\}_{0 \le s \le 1}$ is continously varying.

Suppose ℓ_s bounds a disk in L_s, $\forall\, 0 \leq s < 1$.

We shall prove ℓ_1 bounds a disk in L_1.

Use the metric on M coming from Sullivan's Theorem. L_s are minimal surfaces in M

Let $D_s \subset L_s$ be the minimal area disk in L_s bounding ℓ_s ($\forall\, 0 \leq s < 1$). Then, by isoperineq,

$$\text{area}(D_s) \leq \text{const} \cdot \text{length}(\ell_s)^2$$
$$\leq \text{Const}.$$

$\therefore$ D_s are a family of minimal disks of bounded area.

Sacks-Uhlenbeck (?) compactness $\Rightarrow$

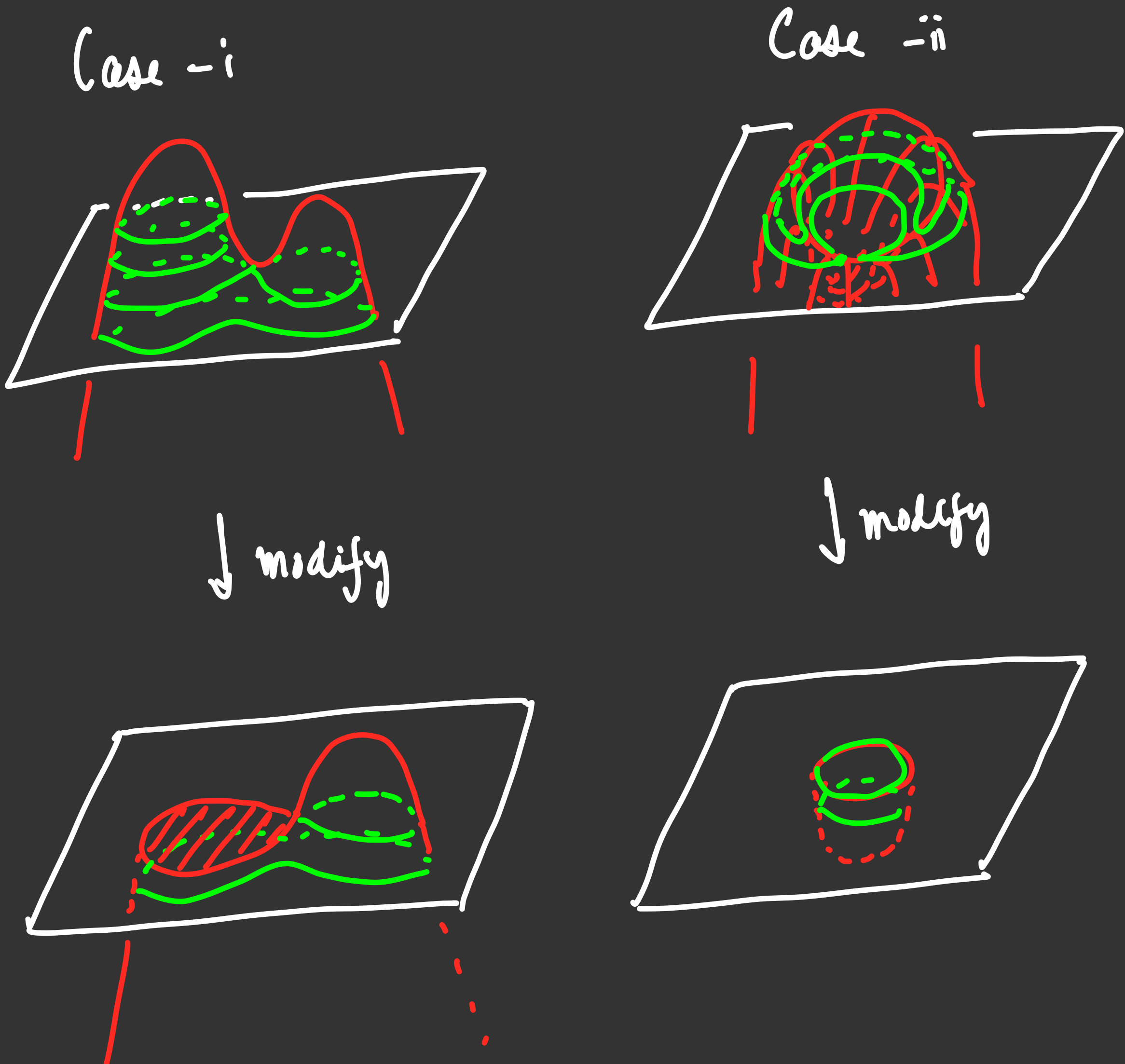
$\mathcal{D}_s$ converge to a disk $\mathcal{D}_1$ with bubbles.

Bubbles can be ignored as we are only interested in pointwise limit.

$\mathcal{D}_1 \subset L_1$ and $\partial\mathcal{D}_1 = \ell_1$. ($\because$ pointwise limit)

So, situation (b) is impossible. $\square$

It remains to deal with situation (a) which <u>will</u> occur in general.



In case (i), push-in the subdisk corresponding to the 1st maximum, smoothen and proceed.

In case (ii), proceed from the minimum.